Two-level Logic Synthesis and Optimization

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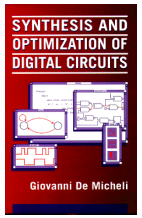


Module 1

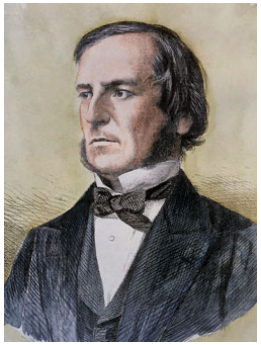
- Objectives
  - Fundamentals of logic synthesis
  - Mathematical formulation
  - Definition of the problems

What is logic synthesis?

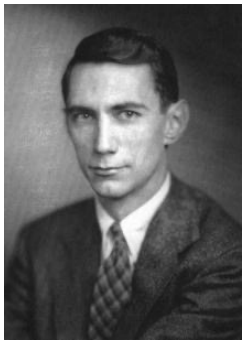
- A *logic-level representation* is:
  - A Boolean function
  - A set of Boolean functions and their dependences
  - A schematic with logic gates
  - A logic-level HDL description
- Logic synthesis is the process of optimizing logic representations with the final goal of:
  - Speeding up a circuit
  - Reducing its area and manufacturing cost
  - Reducing the energy consumption

Logic synthesis has a long history

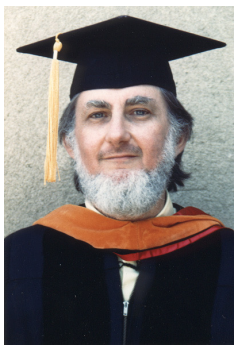
- George Boole:
  - Boolean Algebra



- Claude Shannon
  - Switching Theory



- Modern logic synthesis
  - Ed McCluskey, Robert Brayton, Randy Bryant
    - … and many other noteworthy scientists



Combinational logic design
Background

- **Boolean Algebra**
  - Quintuple $(B, +, ., 0, 1)$
  - Binary Boolean algebra $B = \{ 0, 1 \}$
- **Boolean function**
  - Single output $f : B^n \rightarrow B$
  - Multiple output $f : B^n \rightarrow B^m$
  - Incompletely-specified:
    - *Don't care* symbol: *
    - $f : B^n \rightarrow \{ 0, 1, * \}^m$

Examples of Boolean Algebras

- Algebra of classes:
  - Universal set S – classes are subsets of S
  - 2^S set of subsets of S
  - Quintuple $(2^S, \cup, \cap, \emptyset, S)$
- Arithmetic Boolean algebra
  - n = product of distinct primes
  - D_n = divisors of n
  - Quintuple $(D_n, lcm, gcd, 1, n)$
  - Example n = 30, then ({1,2,3,5,6, 10, 15,30}, lcm, gcd, 1, 30)

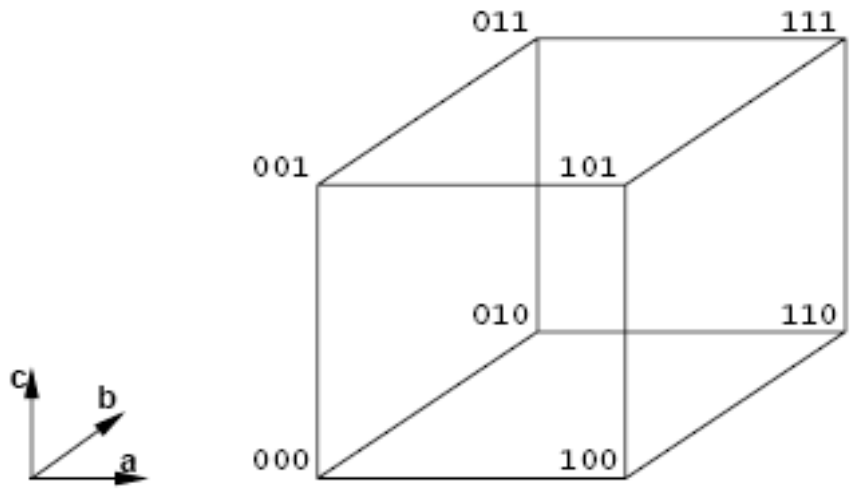
The *don't care* conditions

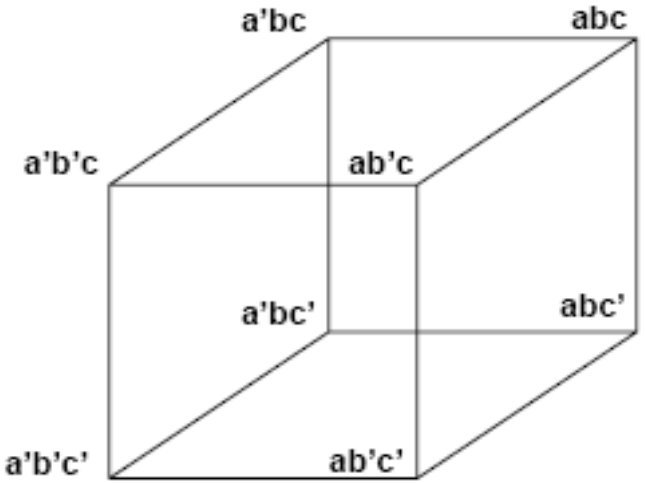
- We do not care about the value of a function
- Related to the environment
  - Input patterns that never occur
  - Input patterns such that some output is never observed
- Very important for synthesis and optimization

Definitions

- **Scalar function:**
  - **ON-set**
    - Subset of the domain such that f is true
  - **OFF-set**
    - Subset of the domain such that f is false
  - **DC-set**
    - Subset of the domain such that f is a *don't care*
- **Multiple-output function:**
  - **ON, OFF, DC-sets defined for each component**

Cubical representation





Definitions

- Boolean variables
- Boolean literals:
  - Variables and their complement
- Product or cube:
  - Product of literals
- Implicant:
  - Product implying a value of the function (usually 1)
  - Hypercube in the Boolean space
- Minterm:
  - Product of all input variables implying a value of the function (usually 1)
  - Vertex in the Boolean space

Tabular representations

- **Truth table**
  - List of all minterms of a function
- **Implicant table** or **cover**
  - List of implicants sufficient to define a function
- **Note:**
  - Implicant tables are smaller in size as compared to truth tables

Example of truth table

- x = ab+a’ c; y = ab+bc+ac

abc	xy
000	00
001	10
010	00
011	11
100	00
101	01
110	11
111	11

Example of an encoded truth table

abc	xy
000	00
001	10
010	00
011	11
100	00
101	01
110	11
111	11

- Using the reverse order:
  - x = 11001010
  - y = 11101000
- Encoded in hexadecimal:
  - x = ca
  - y = e8

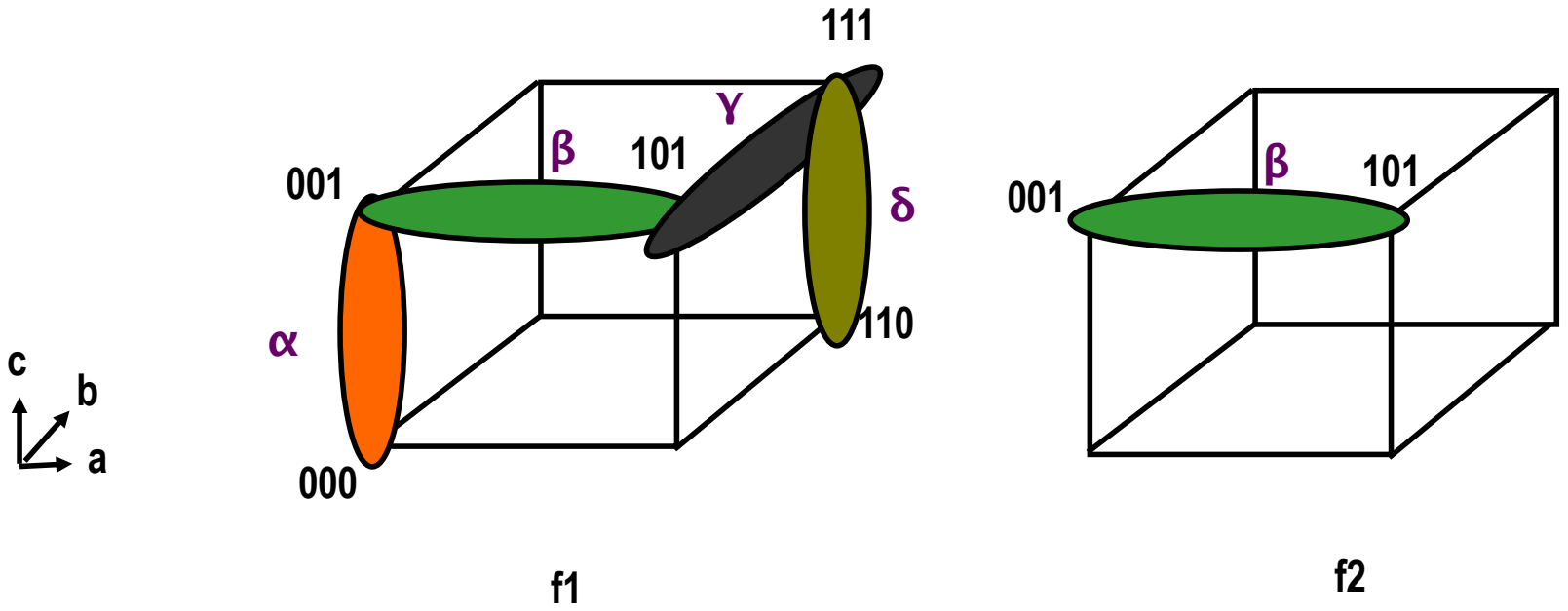
Example of implicant table

- x = ab+a’c; y = ab+bc+ac

abc	xy
001	10
*11	11
101	01
11*	11

Cubical representation of minterms and implicants

- $f_1 = a' b' c' + a' b' c + ab' c + abc + abc'$
- $f_2 = a' b' c + ab' c$



Representations

- Visual representations
  - Cubical notation
  - Karnaugh maps
- Computer-oriented representations
  - Matrices
    - Sparse
    - Various encoding
  - Binary-decision diagrams
    - Address sparsity and efficiency

Module 2

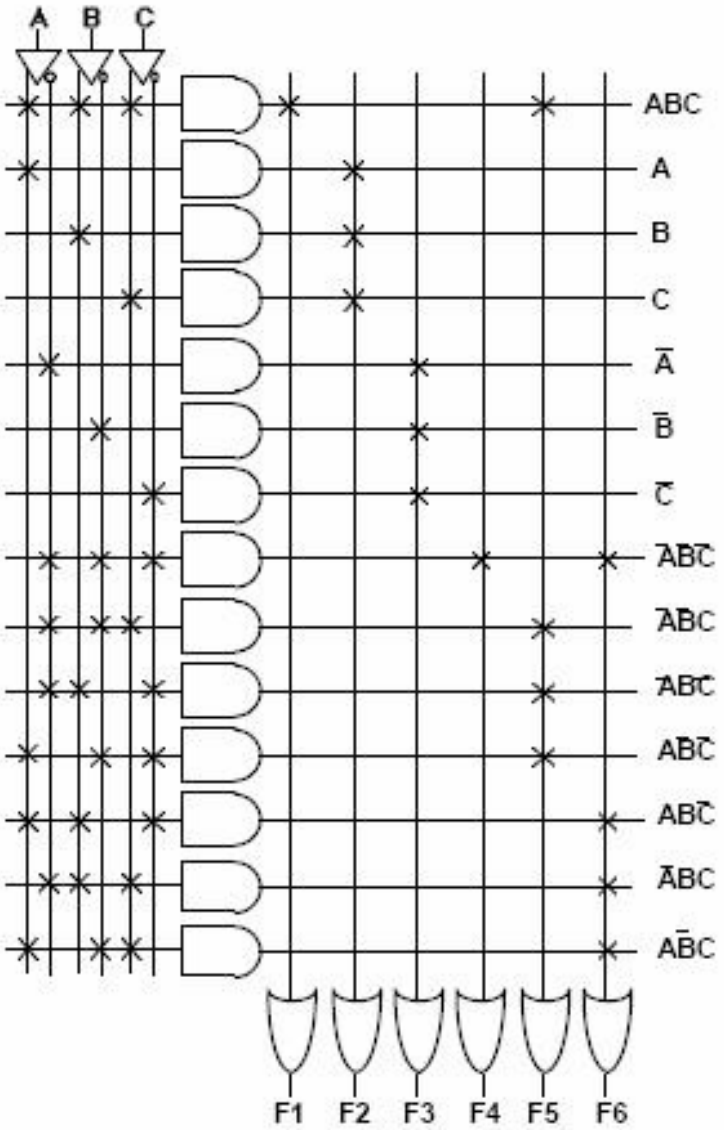
- Objectives
  - Two-level logic optimization
  - Motivation
  - Models
  - Exact algorithms for logic optimization

Two-level logic optimization motivation

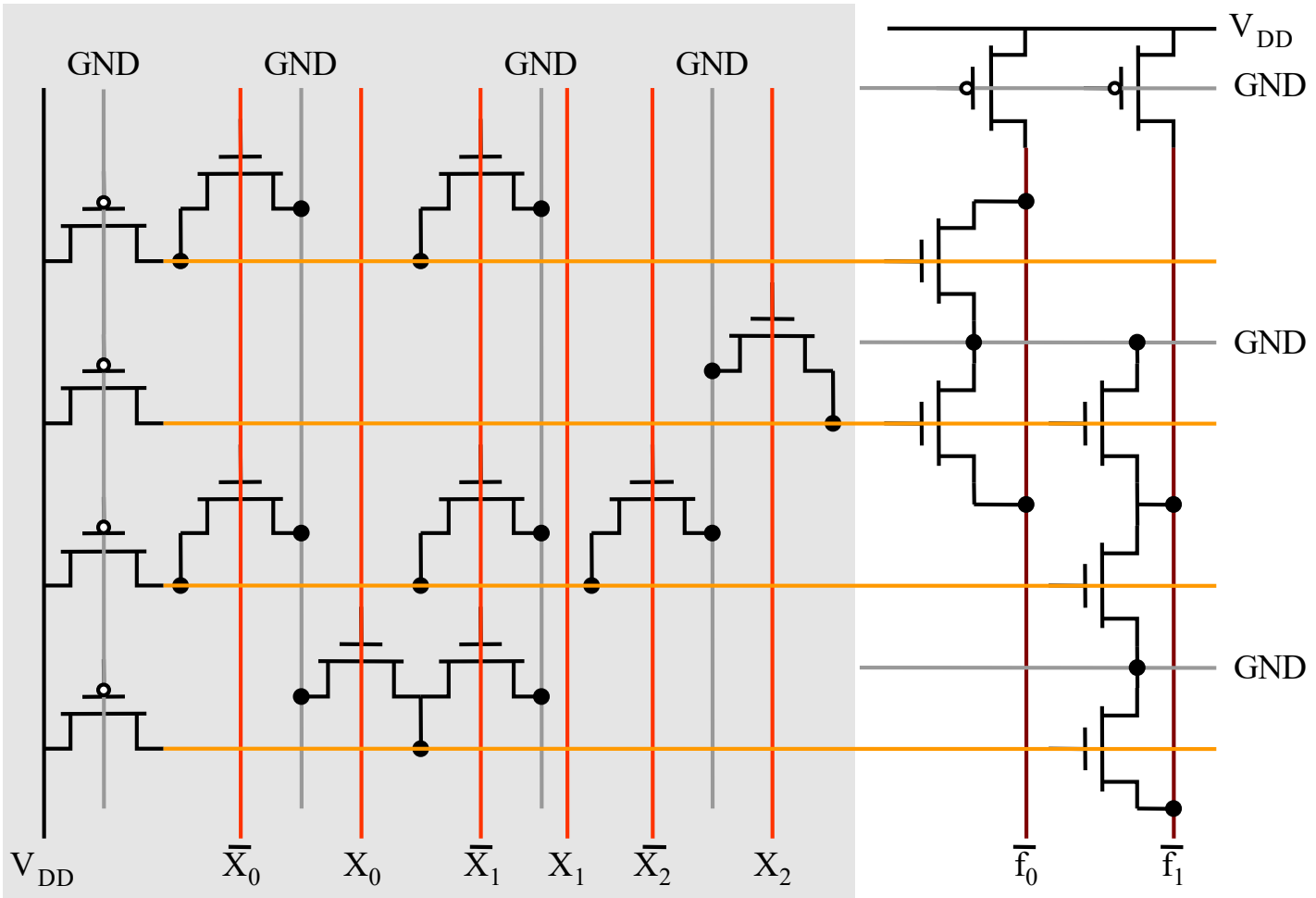
- Reduce size of the representation
- Direct implementation
  - PLAs reduce size and delay
- Other implementation styles
  - Reduce amount of information
  - Simplify local functions and connections

Programmable logic arrays

- Macro-cells with rectangular structure
  - Implement any multi-output function
- Programmable
  - Old technology using fuses
  - Grandfather of FPGAs
- Programmed
  - Layout generated by module generators
  - Fairly popular in the seventies/eighties
- Advantages
  - Simple, predictable timing
- Disadvantages
  - Less flexible than cell-based realization
  - Dynamic operation in CMOS
- Open issue
  - Will PLA structures be useful with new nanotechnologies?



Pseudo NMOS PLA

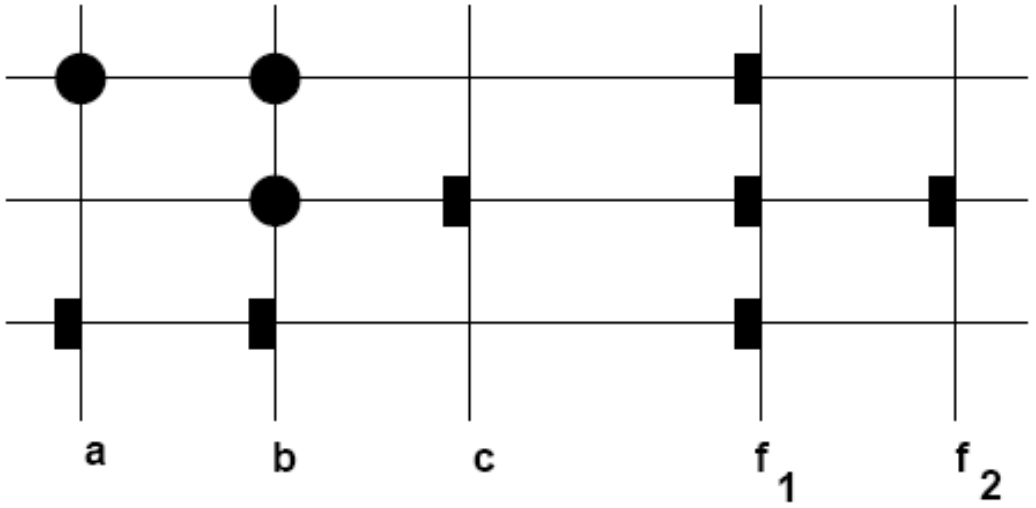


Programmable logic array

- $f_1 = a'b' + b'c + ab$; $f_2 = b'c$

00- 10
-01 11
11- 10

(a)



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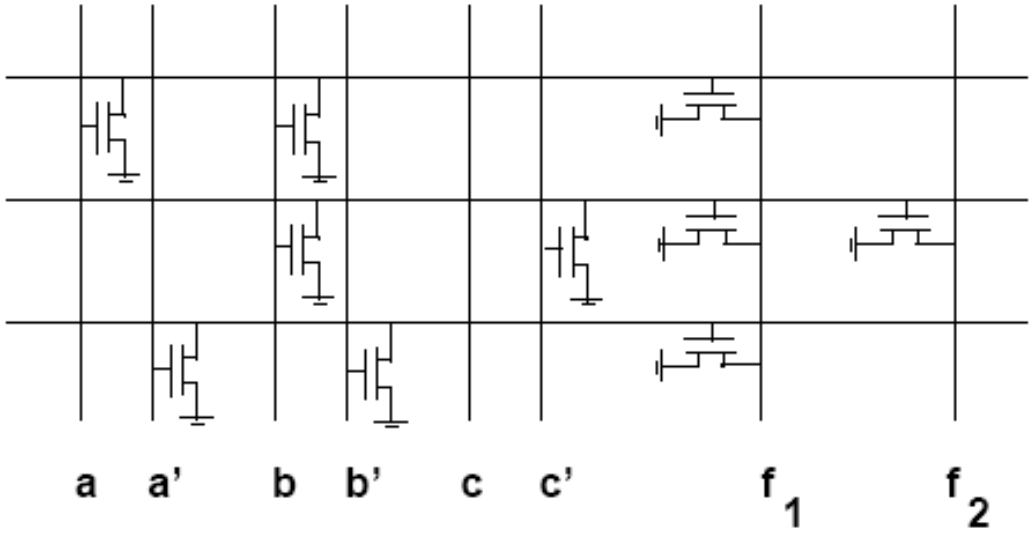


(c)

Two-level minimization

- **Assumptions**
  - Function represented as a Boolean cover
    - Set of implicants (covering all minterms)
  - Primary goal is to reduce the number of implicants
  - All implicants have the same cost
  - Secondary goal is to reduce the number of literals
- **Rationale**
  - Implicants correspond to PLA rows
  - Literals correspond to transistors

Definitions

- **Minimum cover**
  - Cover of a function with minimum number of implicants
  - Global optimum
- **Minimal cover** or **irredundant cover**
  - Cover of the function that is not a proper superset of another cover
  - No implicant can be dropped
  - Local optimum
- **Minimal w.r.to 1-implicant containment**
  - No implicant contained by another one
  - Weak local optimum

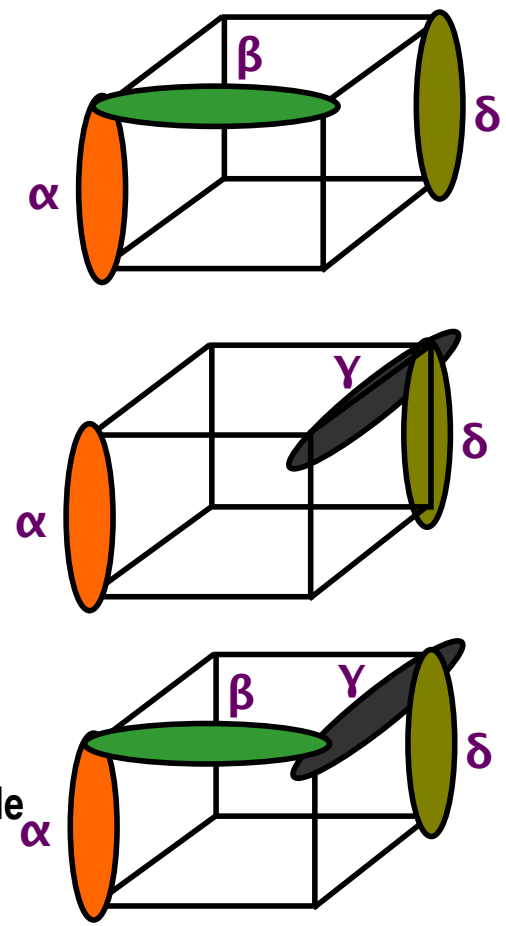
Example

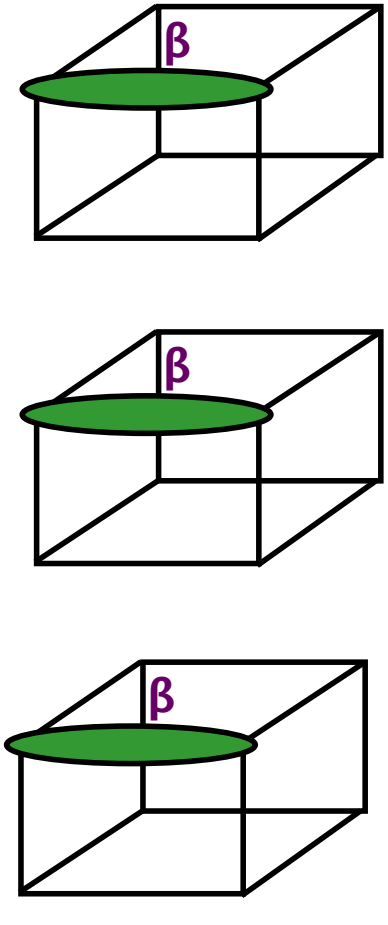
- $f_1 = a' b' c' + a' b' c + ab' c + abc + abc' ; f_2 = a' b' c + ab' c$

Minimum cover

Irredundant cover

Minimal cover w.r. to single implicant containment





f1 f2

Definitions

- **Prime implicant**
  - Implicant not contained by any other implicant
- **Prime cover**
  - Cover of prime implicants
- **Essential prime implicant**
  - There exist some minterm covered only by that prime implicant
  - Needs to be included in the cover

Two-level logic minimization

- **Exact methods**
  - Compute minimum cover
  - Often difficult/impossible for large functions
  - Based on Quine-McCluskey method
- **Heuristic methods**
  - Compute minimal covers (possibly minimum)
  - Large variety of methods and programs
    - MINI, PRESTO, ESPRESSO

Exact logic minimization

- Quine's theorem:
  - There is a minimum cover that is prime
- Consequence
  - Search for minimum cover can be restricted to prime implicants
- Quine-McCluskey method
  - Compute prime implicants
  - Determine minimum cover

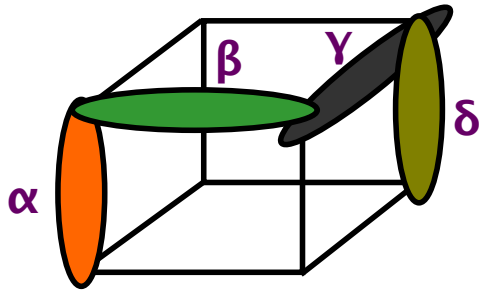
Prime implicant table

- Rows: minterms
- Columns: prime implicants
- Exponential size
  - 2^n minterms
  - Up to $3^n / n$ prime implicants
- Remarks
  - Some functions have much fewer primes
  - Minterms can be grouped together
  - Implicit methods for implicant enumeration

Example

- f = a’ b’ c’ + a’ b’ c + ab’ c +abc +abc’
- Primes:

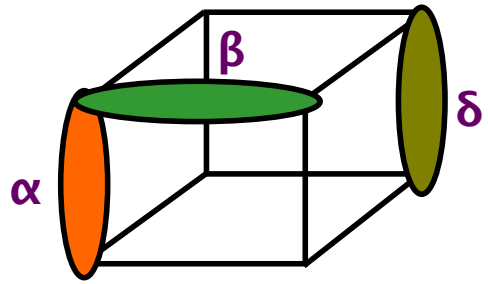
α	00*	1
β	*01	1
γ	1*1	1
δ	11*	1



Prime implicants of f

- Table:

	α	β	γ	δ
000	1	0	0	0
001	1	1	0	0
101	0	1	1	0
111	0	0	1	1
110	0	0	0	1



Minimum cover of f

Minimum cover
early methods

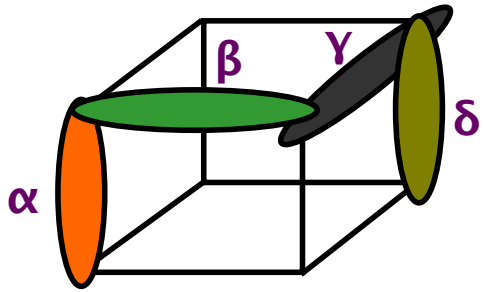
- Reduce table
  - Iteratively identify essentials,
    save them in the cover.
    Remove covered minterms
- Petrick's method
  - Write covering clauses in *pos* form
  - Multiply out pos form into *sop* form
  - Select cube of minimum size
- Remark
  - Multiplying out clauses has exponential cost

Example

- *pos* clauses
  - $(\alpha)\ (\alpha + \beta)\ \ (\beta + \gamma)\ (\gamma + \delta)\ (\delta) = 1$
- *sop* form:
  - $\alpha\beta\delta + \alpha\gamma\delta = 1$
- Solutions:

$\{ \alpha\ \beta\ \delta \}$

$\{ \alpha\ \gamma\ \delta \}$



Matrix representation

- View table as Boolean matrix: A
- Selection Boolean vector for primes: x
- Determine x such that
  - A x ≥ 1
  - Select enough columns to cover all rows
- Minimize norm (1 count) of x

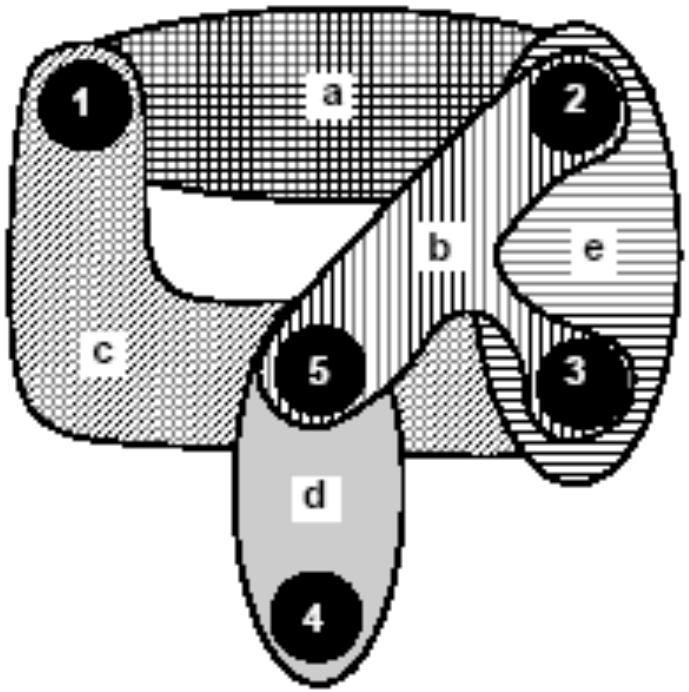
Example

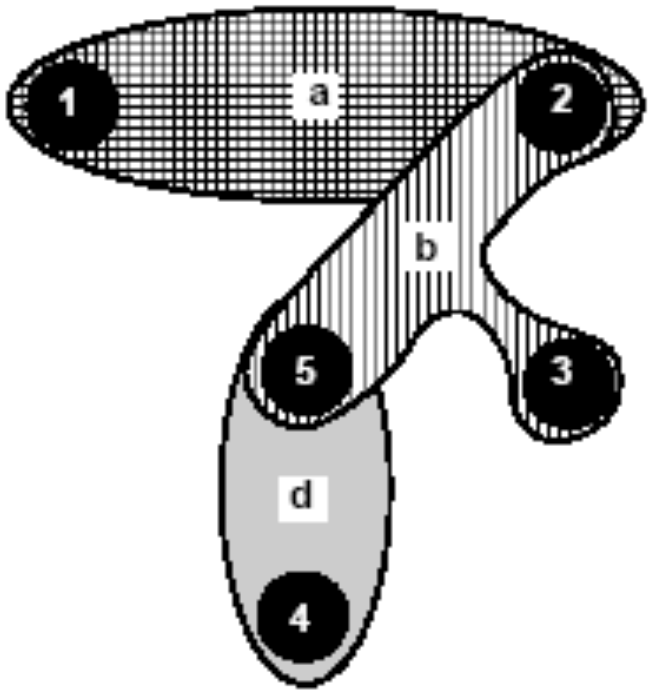
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Covering problem

- Set covering problem:
  - A set S -- minterm set
  - A collection C of subsets (implicant set)
  - Select fewest elements of C to cover S
- Computationally intractable problem
- Exact solution method
  - Branch and bound algorithm
- Several heuristic approximation methods

Example
Edge-cover of a hypergraph

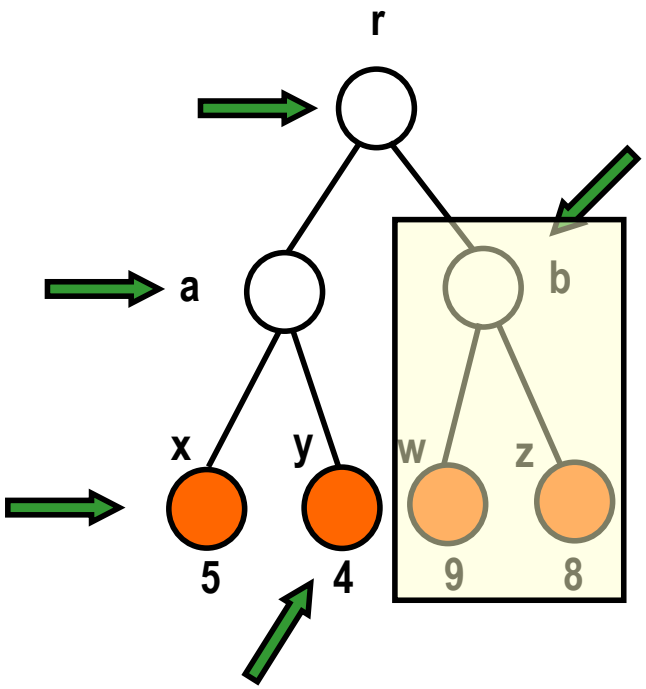




Branch and bound algorithm

- Tree search in the solution space
  - Potentially exponential
- Use bounding function:
  - If the lower bound on the solution cost
    that can be derived from a set of future choices
    exceeds the cost of the best solution seen so far,
    then kill the search
  - Bounding function should be fast to evaluate and accurate
- Good pruning may expedite the search

Example



Bound = 6

Kill sub-tree

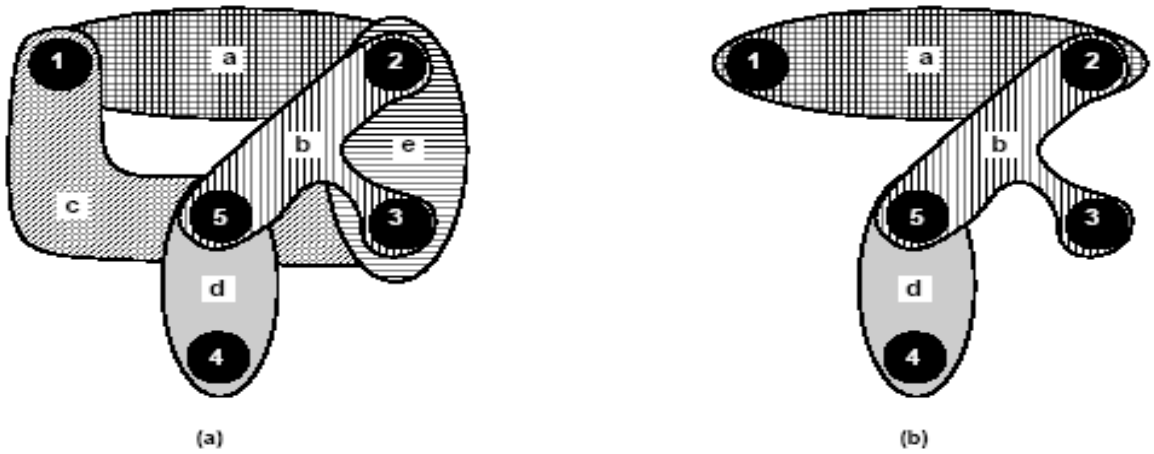
Branch and bound for logic minimization
Reduction strategies

- Use matrix formulation of the problem
- Partitioning:
  - If A is block diagonal:
    - Solve covering problems for the corresponding blocks
- Essentials
  - Column incident to one (or more) rows with single 1
    - Select column
    - Remove covered row(s) from table

Branch and bound for logic minimization
Reduction strategies

- **Column (implicant) dominance:**
  - If $a_{ki} \geq a_{kj}$ for all k
    - Remove column j (dominated)
  - Dominated implicant (j) has its minterms already covered by dominant implicant (i)
- **Row (minterm) dominance:**
  - If $a_{ik} \geq a_{jk}$ for all k
    - Remove row i (dominant)
  - When an implicant covers the dominated minterm, it also covers the dominant one

Example



$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Example

- Fifth row is dominant
- Fourth column is essential
- Fifth column is dominated
- Matrix after reductions:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Branch and bound covering algorithm

```
EXACT_COVER(A,x,b) {
    Reduce matrix A and update corresponding x;
    if (current_estimate ≥ |b|) return (b);
    if (A has no rows) return(x);
    select a branching column c;
    x_c = 1;
    Ã = A after deleting c and rows incident to it;
    x˜ = EXACT_COVER(Ã,x,b);
    if ( | x˜| < |b| )
            b = x˜;
     x_c = 0;
     Ã = A after deleting c;
     x˜ = EXACT_COVER(Ã,x,b);
     if ( | x˜| < |b| )
            b = x˜;
    return(b);
}
```

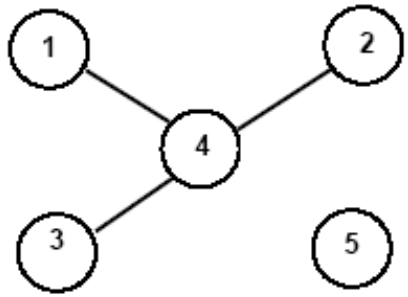
Bounding function

- Estimate lower bound on covers that can be derived from current solution vector **x**
- The sum of the 1s in **x**, plus bound of cover for local **A**
  - Independent set of rows
    - No 1 in the same column
    - Require independent implicants to cover
  - Construct graph to show pairwise independence
  - Find clique number
    - Size of the largest clique
  - Approximation (lower) is acceptable

Example

- Row 4 independent from 1,2,3
- Clique number and bound is 2

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$



Example

- There are no independent rows
  - Clique number is 1 (one vertex)
  - Bound is 1+1= 2
    - Because of the essential already selected

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Example
Branching on the cyclic core

- **Select first column**
  - Recur with $\tilde{A} = [11]$
    - Delete one dominated column
    - Take other column (essential)
  - New cost is 3
- **Exclude first column**
  - Find another solution with cost equal to 3.
  - Discard

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

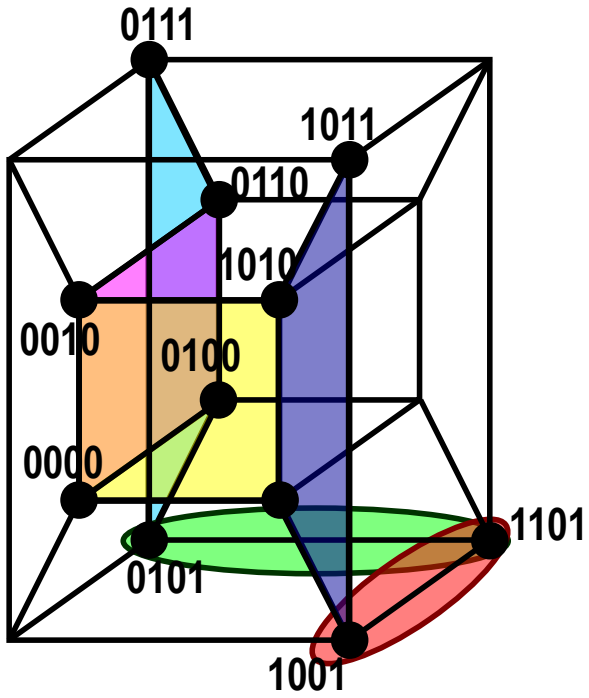
Espresso-exact

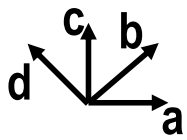
- Exact 2-level logic minimizer
- Exploits iterative reduction and branch and bound algorithm on cyclic core
- Compact implicant table
  - Rows represent groups of minterms covered by the same implicants
- Very efficient
  - Solves most benchmarks

Example

After removing the essentials

	α	β	ε	ζ
0000,0010	1	1	0	0
1101	0	0	1	1





α 0 * * 0 1

β * 0 * 0 1

γ 0 1 * * 1

δ 1 0 * * 1

ε 1 * 0 1 1

ζ * 1 0 1 1

Exact two-level minimization

- There are two main difficulties:
  - Storage of the implicant table
  - Solving the cyclic core
- Implicit representation of prime implicants
  - Methods based on binary decision diagrams
  - Avoid explicit tabulation
- Recent methods make 2-level optimization solve exactly almost all benchmarks
  - Heuristic optimization is just used to achieve solutions faster

Module 3

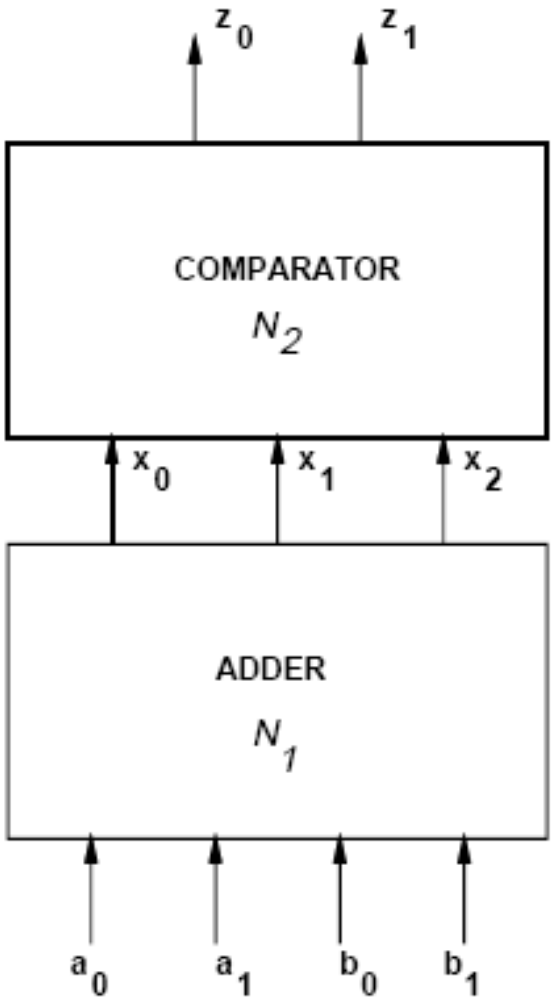
- **Boolean Relations**
  - Motivation of using relations
  - Optimization of realization of Boolean relation
  - Comparisons to two-level optimization

Boolean relations

- Generalization of Boolean functions
- More than one output pattern corresponds to an input pattern
  - Multiple-choice specifications
  - Model inner blocks of multi-level circuits
- Degrees of freedom in finding an implementation
  - More general than *don't care* conditions
- Problem:
  - Given a Boolean relation, find a minimum cover of a compatible Boolean function that can implement the relation

Example

- **Compare:**
  - ▲ a + b > 4 ?
  - ▲ a + b < 3 ?



Example

a_1	a_0	b_1	b_0	x
0	0	0	0	{ 000, 001, 010 }
0	0	0	1	{ 000, 001, 010 }
0	0	1	0	{ 000, 001, 010 }
0	1	0	0	{ 000, 001, 010 }
1	0	0	0	{ 000, 001, 010 }
0	1	0	1	{ 000, 001, 010 }
0	0	1	1	{ 011, 100 }
0	1	1	0	{ 011, 100 }
1	0	0	1	{ 011, 100 }
1	0	1	0	{ 011, 100 }
1	1	0	0	{ 011, 100 }
0	1	1	1	{ 011, 100 }
1	1	0	1	{ 011, 100 }
1	0	1	1	{ 101, 110, 111 }
1	1	1	0	{ 101, 110, 111 }
1	1	1	1	{ 101, 110, 111 }

Note that output 111 cannot be a+b but can be considered as a *don't care*

Example

- Circuit is no longer an adder

a_1	a_0	b_1	b_0	x
0	*	1	*	010
1	*	0	*	010
1	*	1	*	100
*	*	*	1	001
*	1	*	*	001

Minimization of Boolean relations

- Since there are many possible output values (for any input), there are many logic functions implementing the relation
  - Compatible functions
- Problem
  - Find a minimum compatible function
- Do not enumerate all compatible functions
  - Compute the primes of the compatible functions
    - C-primes
  - Derive a logic cover from the c-primes

Example

- Boolean relation:
- 4 compatible functions
- Assume c-primes include:

  ε * 1 1 01

  ζ 1 * 1 10

- To cover minterm 111:
  - Take both primes or none
  - ε ζ + ε' ζ'
  - Petrick's function is binate

abc	xy
000	00
001	00
010	00
011	10
100	00
101	01
110	{00,11}
111	{00,11}

Minimizing Boolean relations

- Minimizing Boolean *relations* is more complex
  - As compared to minimizing *Boolean* ***functions***
- In classic Boolean minimization we just need to select enough implicants to cover the minterms
  - Covering clause is unate in all variables
  - Any additional implicant does not hurt
- In Boolean relation optimization, we need to pick implicants to realize a compatible function
  - Some implicants cannot be taken together
  - Choosing an implicant may imply rejecting an other
  - Covering clause is binate (implicant mutual exclusion)

Solving binate covering

- Binate cover can be solved with branch and bound
  - In practice much more difficult to solve, because it is harder to bound effectively
- Binate cover can be reduced to min-cost SAT
  - SAT solvers can be used
- Binate cover can be also modeled by BDDs
- Several approximation algorithms for binate cover

Summary: Boolean Relations

- Generalization of Boolean functions
  - More degrees of freedom than *don't care* sets
- Useful to represent multiple choices
- Useful to model internals of logic networks
- More general formalism
  - But computationally-intensive solution method